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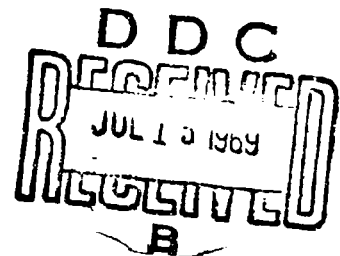
SOME REMARKS ON CIRCULAR PROBABLE ERROR  
AND OTHER STATISTICS OF TWO-DIMENSIONAL  
DISTRIBUTIONS

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# ABSTRACT

A definition is given of Circular Probable Error (CEP) as used in papers produced by the Seventh Air Force Directorate of Tactical Analysis. Other statistical parameters used in publications of that office are also discussed, as well as orientations used in analyses of navigation and weapon-delivery errors. A graphical method is presented for transforming from standard deviations on two orthogonal axes to CEP. (U)

1. The term "Circular Probable Error" refers to a statistic used in analysis of errors distributed in two dimensions, in particular, navigational and weapon-delivery errors; in these military contexts it is usually referred to as "CEP." Unfortunately, the term is ambiguous in that different users define it differently, both operationally and in the abstract: To some it is the median radial error, i.e., the radius of the circle centered at the target containing half the impacts. (For convenience, weapon-delivery terms will be used, but with appropriate substitutions the discussion is also applicable to other two-dimensional distributions.) To others CEP is the radius of a circle centered at the population (or sample) mean and containing half the population (sample). To still others CEP means a circle centered at the population (or sample) mean with radius defined by some arbitrary function of the linear standard deviations, usually on the assumption of bivariate or circular normality. These varying definitions may yield significantly different results, particularly with small samples, and since writers frequently omit to state the definitions they are using, one reading a report often really does not know what is meant. It would seem that a similar but better statistic would be Standard Deviation, which has a well-known common definition independent of any assumption as to the form of the distribution; however, it must be admitted that the term CEP has a certain concreteness which Standard Deviation lacks, so that CEP is not likely to disappear from the literature, especially that addressed to operational personnel rather than to analysts. The purpose of this Note is to define what is meant by CEP as used in papers produced in the Seventh Air Force Directorate of Tactical Analysis (DOA), to present a graphical method of determining CEP from a set of data, and to discuss certain related matters.

2. For analytic purposes, the statistic actually used in DOA is the Standard Deviation in one dimension,  $\sigma$ , or rather the sample Standard Deviation  $s$  as the best estimate of  $\sigma$ . However, for publication, an equivalent CEP is computed on the basis of normal circular or bivariate distribution. The definition of  $s$  is the usual one:

$$s = \left[ \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \right]^{1/2}$$

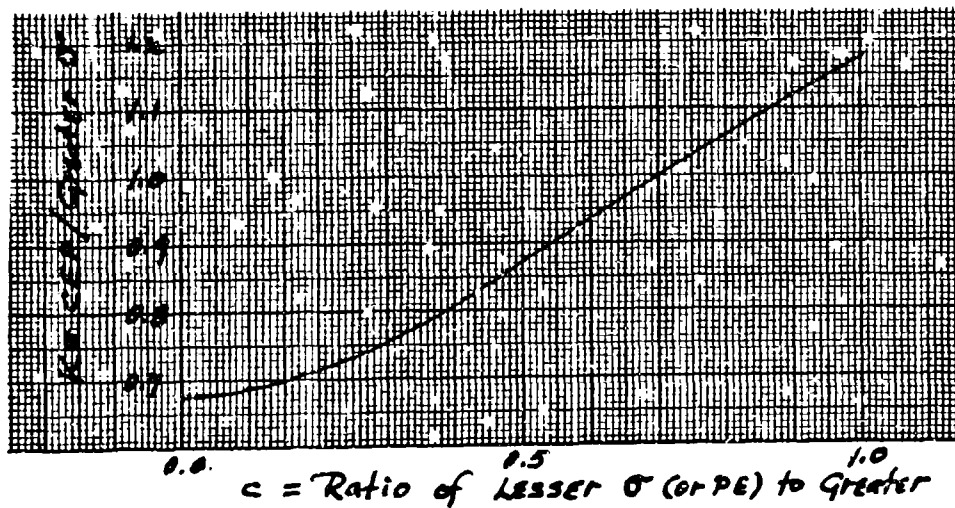
which can be reduced algebraically to several more usable forms, such as

$$s = \left\{ \frac{1}{n(n-1)} \left[ n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2 \right] \right\}^{1/2}$$

In the above,  $\bar{x}$  is the sample mean of  $n$  observations (in this case errors).

In practice, errors are usually resolved with respect to a pair of orthogonal axes (see paragraph 4 below) and  $\bar{x}$  and  $s$  are computed for each; analytic manipulations may then be performed on these values. To prepare for publication of results, CEP is then determined by means of Figure 1: The lesser  $s$  is divided by the greater to obtain a number  $c = s_1/s_2$  such

FIGURE 1

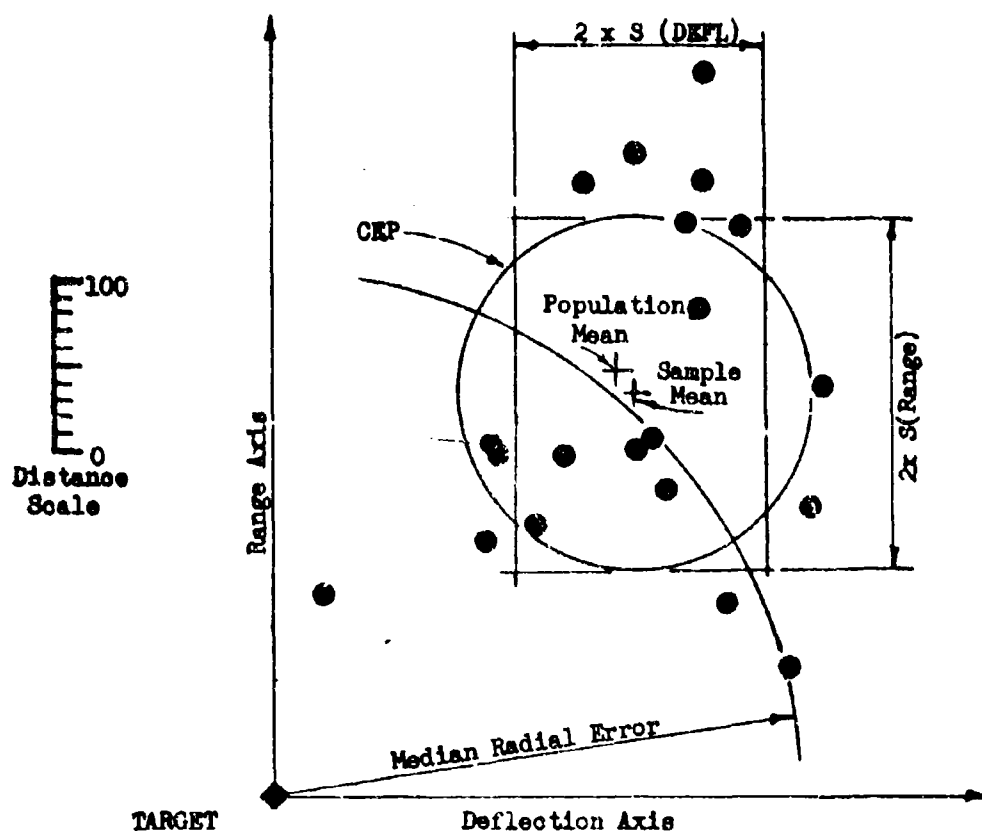


that  $0 \leq c \leq 1$ . With  $c$  one enters the  $c$  (horizontal) axis of Figure 1 and reads the corresponding ordinate, a number  $K$  such that  $0.6745 \leq K \leq 1.1774$ , which is then multiplied by  $s_2$  (i.e., the greater of  $s_1$  and  $s_2$ ) to obtain CEP.

3. Figure 1 was constructed from reference 1, Table II.6, "Normal Distribution Circular Error Probabilities." Because of copyright the table is not reproduced here, but it is described as a tabulation of " $P(K, c)$  = the probability that a point falls inside a circle whose center is at the origin and whose radius is  $K$  times the larger standard deviation,  $c$  being the ratio of the smaller standard deviation to the larger standard deviation." To construct Figure 1, for each value of  $c$  tabulated (0.0 to 1.0 at 0.1 intervals) values of  $P$  were plotted as a function of  $K$  in the neighborhood of  $P=0.5$ . By graphic interpolation the single  $K$  was determined for each  $c$  such that  $P=0.5$ , and Figure 1 is the plot of the number-pairs  $(K, c)$  such that  $P(K, c) = 0.5$ . Implicit in construction of the table are the assumptions that the origin is defined by the orthogonal mean errors, and the orthogonal errors are normally distributed. That in a particular instance the orthogonal errors may not be in fact normally distributed does not affect our definition of CEP. Figure 2 is an example showing relationships of standard deviations to CEP so determined, as well as other statistics.

4. The matter of selecting proper coordinate axes needs some comment: Usually the errors are tabulated in azimuthal coordinates (impact so far from target at such-and-such azimuth); thus they are oriented with respect to the meridian. However, frequently a more significant orientation exists:

FIGURE 2 - CEP AND OTHER PARAMETERS



Parameters associated with a sample of 20 observations are shown. The sample was drawn from a population known to be bivariate normal (constructed from Table XII.5 of reference 1). Here is a comparison of some parameters estimated from the sample with known values for the population:

	<u>SAMPLE</u>	<u>POPULATION</u>
Mean error, range	235	250
Mean error, deflection	213	200
Standard Deviation, range	100	125
Standard Deviation, deflection	74	100
CEP	102	132

aircraft attack heading for example, or the radius vector from a ground radar control site to a target. The orientation selected may be very important in revealing or concealing systematic errors: An error population which appears to have a large dispersion ( $\sigma$  or CEP) but relatively insignificant bias under one orientation may, under another, exhibit a much smaller dispersion but a large bias, indicative of systematic (and thus presumably correctible) rather than random causes. It should be noted that orientation does not enter into Median Radial Error nor into CEA, and these statistics reveal nothing as to bias.

5. Another statistic occasionally used in DOA publications is Median Radial Error (MRE); i.e., the radius of the circle centered at the target and enclosing half the impacts. To determine it, radial errors (distances from target to impact) are ordered by magnitude and the mid-value is taken (for an even number of impacts the arithmetic mean of the two mid-values is taken):

Example 1: Errors are 1, 2, 3, 4, 500;

MRE = 3, the mid-value.

Example 2: Errors are 101, 102, 103, 104;

MRE =  $\frac{102 + 103}{2} = 102.5$ , average of the two mid-values.

Example 3: Errors are 1, 2, 2, 2, 10;

MRE = 2, the mid-value.

Example 4: Errors are 1, 2, 2, 2;

MRE =  $\frac{2+2}{2} = 2$ , average of the two mid-values.

6. The last statistic which may (rarely) appear in DOAR papers is Average Circular Error (CEA), defined as the arithmetic mean (average) of the magnitudes of the radial errors (distances from target to impact points).

Example: Errors are 1, 2, 3, 4, 500;

$$CEA = \frac{1+2+3+4+500}{5} = \frac{510}{5} = 102.$$

7. To summarise, the important DOA statistics are the sample Standard Deviation  $s$ , and the CEP derived from the  $s$ -values for range and for deflection (or other orthogonal coordinates) as described in paragraph 2 above.

8. A rather complete discussion of the statistics associated with normal distributions in two and three dimensions is contained in reference 2.

#### REFERENCES

1. Beyer, W. H. (ed), CRC Handbook of Tables for Probability and Statistics, the Chemical Rubber Co., Cleveland, Ohio, 1966.
2. AMP Note 13, Notes on Parameters of Probability Distributions, the Applied Mathematics Panel, National Defense Research Council, 23 June 1944, available from the Defense Documentation Center (DDC), Cameron Station, Alexandria, Va 22314, under DDC Accession Number ATI-21233.

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